2021

PHYSICS — HONOURS

(Syllabus : 2019-2020)

Paper : CC-8

(Mathematical Physics III)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer ***question no.* 1** and ***any four*** questions from the rest.

1. Answer ***any five*** questions : 2×5

(a) Find the principal value of i^i, where $i = \sqrt{-1}$.

(b) Find the residue of $f(z) = ze^{\frac{1}{z^2}}$ at its pole.

(c) Consider three functions : (i) $f_1(z) = |z|^3$, (ii) $f_2(z) = \sinh z$, (iii) $f_3(z) = (1 + z^*)^{10}$, where $z^* = x - iy$. State with reason which of the following functions is / are not analytic.

(d) Find the equation of motion for the Lagrangian $L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 + 2q\dot{q} + 3q^2\dot{q}$.

(e) Show that the conjugate momentum corresponding to a cyclic variable in the Lagrangian is conserved.

(f) Lifetime of muon in its rest frame is 2×10^{-6} s. How, then, a muon produced at a height of 4 km can reach the surface of the earth?

(g) Consider two events A and B in an inertial frame S with four coordinates $(ct, x, y, z) = (13, 12, 5, 0)$ and (0, 0, 3, 4) respectively. In another inertial frame S' moving with a velocity $\frac{c}{2}$ along the common x-axis. What should be the separation ds^2 between A and B?
[Use the metric convention (1, –1, –1, –1)]

2. (a) Find the Laurent series of

$$f(z) = \frac{1}{z(z-2)^3}$$

about the singularities $z = 0$ and $z = 2$ separately. From the series, verify that $z = 0$ is a pole of order 1 and $z = 2$ is a pole of order 3. Also find the residue of $f(z)$ at each pole.

(b) Given real part of the analytic function $u = e^{-x}(x \sin y - y \cos y)$, find $f(z)$. (3+2+2)+3

Please Turn Over

3. (a) Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

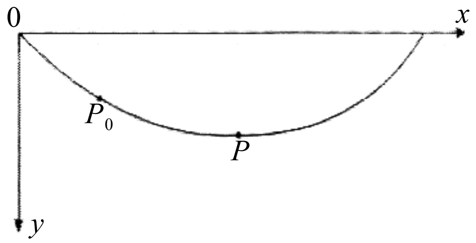
(b) Calculate $\oint_{|z|=1} \frac{\sin z}{\left(z^2 - \frac{\pi^2}{16}\right)} dz$

(c) Find the nature of singularity of $f(z) = \frac{\sinh z}{z^4}$. 5+3+2

4. (a) A particle is constrained to move on the surface of a sphere. What are the equations of constraint for this system?

(b) Consider a single loop of the cycloid having a fixed value of a as shown in the figure. A car released from rest at any point P_0 anywhere on the track between O and the lowest point P, that is, P_0 has a parameter $0 < \theta_0 < \pi$. Take

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$



Show that the time T for the car to slide from P_0 to P is given by the integral

$$T(P_0 \to P) = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1-\cos\theta}{\cos\theta_0 - \cos\theta}}\, d\theta$$

Prove that this time T is equal to $\pi\sqrt{a/g}$, which is independent of the position P_0. [*Hint : You might need to substitute* $\theta = \pi - 2\alpha$ *to calculate the integral easily.*] 2+(3+5)

5. (a) A particle of mass m is moving on the inner surface of a paraboloid of revolution $x^2 + y^2 = 4z$ under gravity along z direction. Construct the Lagrangian and hence find the equations of motion.

(b) Is there any cyclic coordinate in part (a)? Find the conserved momentum.

(c) $L(x,\dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \lambda x^3 + \mu x \dot{x}^2$ where ω, λ, μ are constants.

(i) Find conjugate momentum

(ii) Find the Hamiltonian

(iii) Is energy conserved in this system? 4+2+(1+2+1)

6. (a) A rod of proper length L_0 is at rest in an inertial frame S'. The rod is inclined at an angle θ' with respect to the x'-axis of S'. If S' moves with a uniform velocity v relative to another inertial frame S along the common x-axis, show that

(i) the length of the rod in S-frame is

$$L = L_0\left(\frac{\cos^2\theta'}{\gamma^2} + \sin^2\theta'\right)^{\frac{1}{2}}$$

(ii) the angle of inclination of the rod in S-frame is

$\theta = \tan^{-1}(\gamma \tan\theta')$,

where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$.

(b) A meson of rest mass π comes to rest and disintegrates into a muon of rest mass μ and a neutrino of zero rest mass. Show that the kinetic energy of the muon (i.e. without the rest mass energy) is

$$T = \frac{(\pi - \mu)^2 c^2}{2\pi}$$

(2+2)+6

7. Consider 4-momentum $p^\mu = \left(\frac{E}{C}, \vec{p}\right)$ is an inertial frame S.

(a) Write down the, Lorentz transformation equations of p^μ in an inertial frame S', moving along common x-axis w.r.t. S.

(b) Show that for any 4-vector A^μ is invariant under Lorentz transformation.

(c) Find $P^\mu P_\mu$ in the rest frame of the particle.

(d) Show that 4-force and 4-momentum are orthogonal to each other. 3+3+2+2

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(d) Give reason why the fourier transform of e^x does not exist.

(e) Two unbiased dice are rolled. Find the probability that the sum is equal to 5.

(f) Lifetime of muon in its rest frame is 2×10^{-6} s. How, then, a muon produced at a height of 4 km can reach the surface of the earth?

(g) Consider two events A and B in an inertial frame S with four coordinates $(ct, x, y, z) = (13, 12, 5, 0)$ and $(0, 0, 3, 4)$ respectively. In another inertial frame S' moving with a velocity $\frac{c}{2}$ along the common x-axis. What should be the separation ds^2 between A and B?

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3. (a) Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

(b) Calculate $\oint_{|z|=1} \frac{\sin z}{\left(z^2 - \frac{\pi^2}{16}\right)} dz$

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4. (a) Find the exponential Fourier transform of $e^{-|x|}$ and hence find the value of the integral

$$\int_0^{\infty} \frac{\cos \alpha x}{\alpha^2+1} d\alpha$$

(b) Using Fourier transform, solve the equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < \infty,\ t > 0 \quad (k \text{ is constant})$$

subject to conditions

(i) $u(0, t) = 0$ $\quad t > 0$

(ii) $u(x, 0) = e^{-x}$ $\quad x > 0$

(iii) u and $\frac{\partial u}{\partial t}$ both tend to zero as $x \to \pm\infty$. (3+2)+5

5. (a) The *standard deviation* is defined as

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \langle x\rangle\right)^2}$$

where x_i are the values of some random variable x and $\langle\ \rangle$ denotes the mean value. Show that

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2.$$

Suppose from a measurement we get $x_1 = 4$, $x_2 = 0$, $x_3 = -1$, $x_4 = 2$, $x_5 = 5$. Calculate its standard deviation.

(b) The *Gaussian* (*normal*) *distribution* is defined by the probability density

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{[x-\mu]^2}{2\sigma^2}\right),\quad -\infty<x<+\infty$$

(i) Show that it is properly normalized.

(ii) Show that the mean and variance of this distribution are μ and σ, respectively.

(iii) Roughly plot the distribution with $\mu=0$ and $\sigma^2=0.1,\ 1.0,\ 10$ in a same figure.

(1½+1½)+(2+3+2)

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