2020

PHYSICS — HONOURS

Paper : DSE-A-1

(Advanced Mathematical Methods Theory)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer ***question nos.*** **1** & **2**, and ***any four*** questions from the rest (**Q.3** to **Q.8**).

1. Answer ***any five*** questions : 2×5

(a) If P_1 and P_2 are two projection operators, then under what condition is $P_1 + P_2$ also behaves like a projection operator?

(b) Show that any two vectors $|V_1\rangle$ and $|V_2\rangle$ $(\neq 0)$ that are orthogonal to each other are linearly independent.

(c) Use transformation property of cartesian tensors to establish that every contraction reduces the rank of a tensor by 2.

(d) Show that δ^i_j is an isotropic tensor.

(e) If $|q\rangle$ is any eigenstate of operator $\hat{Q}$ such that $\hat{Q}|q\rangle = q|q\rangle$ and suppose that $\hat{C}$ is another operator with $\hat{C}|q\rangle = |-q\rangle$, show that $\hat{C}\hat{Q} = -\hat{Q}\hat{C}$.

(f) The line element in metric form is given by $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$. Use Quotient law to argue that $g_{\mu\nu}$ is a covariant tensor of rank two.

(g) Let e_1, e_2, e_3 be the generators of a three dimensional Lie algebra with the commutation relation $[e_1, e_2] = 0$, $[e_2, e_3] = e_1 + e_2$; $[e_3, e_1] = -e_1$. Find $[e_1, [e_2, e_3]] + [e_3, [e_1, e_2]]$.

2. Answer ***any three*** questions :

(a) Given the set of vectors $u_1 = (2, -1, 0)$, $u_2 = (1, 0, -1)$, $u_3 = (3, 7, -1)$. Use Gram-Schmidt orthogonalization procedure, with the standard Euclidean inner product, to find an orthonormal set. 5

(b) Prove the vector identity $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})\,(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})\,(\vec{B}\cdot\vec{C})$ using $\epsilon_{ijk}\,\epsilon_{pqk} = \delta_{ip}\,\delta_{jq} - \delta_{iq}\,\delta_{jp}$. 5

(c) An antisymmetric tensor $F_{\mu\nu}$ satisfies $\partial_\mu F^{\mu\nu} = j^\nu$.

(i) Show that $\partial_\nu j^\nu = 0$.

(ii) Find out the number of independent components of $F_{\mu\nu}$ in 4 dimensions.

(iii) Show that for any mixed tensor T^μ_ν, T^μ_μ is a scalar. 2+1+2

(d) (i) Show that all $(n\times n)$ unitary matrices form a group under multiplication.

(ii) Show that all such matrices with determinant 1 forms a subgroup. 3+2

(e) Show that all integers including zero form a group under addition. Write down the identity element. Do they form a group under subtraction? Justify. 2+1+2

3. Show that the matrix $A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$

(a) is Hermitian.

(b) Find out its eigenvalues.

(c) Find out eigenvectors and show that they are orthogonal.

(d) Denoting the eigenvectors $|1\rangle$ and $|2\rangle$, show that $|1\rangle\langle 1| + |2\rangle\langle 2| = \mathbb{I}$. 1+2+(3+1)+3

4. Starting with a vector space consisting of functions
$f_0(x) = 1,\ f_1(x) = x,\ f_2(x) = x^2, \ldots,\ f_n(x) = x^n$ and defining the scalar product as

$$\langle f_m \mid f_n \rangle \equiv \int_{-1}^{1} f_m(x) f_n(x)\, dx$$

(a) Show that these functions are not orthogonal to each other.

(b) Starting from $f_0(x)$ and $f_1(x)$, use Gram-Schmidt method to construct first three polynomials $P_0(x)$, $P_1(x)$ and $P_2(x)$.

(c) Normalize these polynomials such that $P_n(x = 1) = 1$.

(d) Show that $P_2(x)$ is an eigenfunction of the operator $\left(1 - x^2\right)\dfrac{d^2}{dx^2} - 2x\dfrac{d}{dx}$. Find the eigenvalue.

2+4+2+2

5. Consider two vectors (cartesian) $\vec{A}$ and $\vec{B}$ in 3D. For an anticlockwise rotation about the z-axis by an angle θ.

(a) How do the components of the vectors change?

(b) Show that for a norm-preserving rotation $\sum_i A_i B_i$ is a scalar, using transformation properties.

(c) Define :

$$C_1 \equiv A_2B_3 - A_3B_2$$
$$C_2 \equiv A_3B_1 - A_1B_3$$
$$C_3 \equiv A_1B_2 - A_2B_1$$

Show that under this rotation C_1, C_2, C_3 transform like the components of a vector.

(d) Identify $\vec{C}$ in terms of $\vec{A}$ and $\vec{B}$. 2+2+5+1

6. (a) Write down the Moment of Inertia tensor I_{ij} explaining each term.

(b) Show that I_{ij} is symmetric.

(c) Show that I_{ij} transforms as a 2nd rank tensor.

(d) Construct the inertia matrix for a system of three point masses of 1 unit, 2 units and 1 unit placed at (1, 1, –2), (–1, –1, 0) and (1, 1, 2) respectively. 2+1+3+4

7. (a) Construct the group multiplication table for the set of elements $\{1, i, -1, -i\}$. Find out the identity element. Find the inverse of the element i and –1.

(b) Consider $\{1, -1\}$. Show that they form a subgroup of the above group.

(c) Consider two matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Show that the map $1 \rightarrow A$ and $-1 \rightarrow B$ is a homomorphism. (3+1+2)+2+2

8. (a) Write down the generators for $SU(2)$ group in fundamental representation.

(b) Find out the non-zero structure constants.

(c) Find out the adjoint representation of $SU(2)$ group. 2+3+5